THE GLOBAL DECLINE OF THE LABOR SHARE

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Research Area
Macroeconomics, Labor Economics

Publication
- “The Global Rise of Corporate Saving.” Journal of Monetary Economics, 89, 1-19, August 2017. Lead Article. (With Peter Chen and Brent Neiman.)
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Professor, Booth School of Business, University of Chicago

Research Area
Macroeconomics, International Finance

Publication
Content:

- Introduction
- Model
- Estimation of Elasticity
- Conclusion and Implication
Introduction

\[ Y = AK^\alpha L^{1-\alpha} \]

\[ \frac{E_L}{E} = 1 - \alpha \]

\( E_L \): Expenditure on Labor, \( E \): Total Expenditure

**Motivation:**

- A significant decline of global labor share since 1980s
- A significant decline in relative price of investment goods.
Trends in Labor Share

Data:

• Country-level statistics in the corporate sector
• Sources: BEA, UN, OECD, KLEMS
• Time Period: 1975-2012
• Country: 59 countries that have at least 15 years of data
Trends in Labor Share

Figure I
Declining Global Labor Share
Trends in Labor Share

United States

Japan

China

Germany

Figure II
Declining Labor Share for the Largest Countries
Trends in Labor Share

Figure III
Estimated Trends in Country Labor Shares
Trends in Labor Share

Figure IV
Estimated Trends in U.S. State Labor Shares
Trends in Labor Share

Figure V
Estimated Trends in Industry Labor Shares
Trends in Labor Share

Labor share change:
• Changing size of industries with different level of labor share
• Changes in labor shares within industry

\[
\Delta s_{Li} = \sum_k \bar{\omega}_{i,k} \Delta s_{Li,k} + \sum_k s_{Li,k} \Delta \omega_{i,k} ,
\]

\[
\begin{align*}
\text{Within-Industry} & \quad |\text{Between-Industry} \\
\text{Within-Industry} & \quad |\text{Between-Industry}
\end{align*}
\]

\(\omega_{i,k}\): industry k's share in country i's value added

\(S_{Li,k}\): labor share in country i's industry k
Trends in Labor Share

**Figure VI**
Trends in Relative Price of Investment Goods

Data:

• Country-level

• Sources: PWT, WDI, KLEMS

• Time Period: 1950-2012
Trends in Relative Price of Investment Goods

Figure VII
Declining Global Price of Investment Goods
A Model of The Labor Share
A. Final Consumption Good

\[ C_t = \left( \int_0^1 c_t(z)^{\frac{\varepsilon_i - 1}{\varepsilon_i}} \, dz \right)^{\frac{1}{\varepsilon_i - 1}} \]

\[ P_t^c = \left( \int_0^1 p_t(z)^{1-\varepsilon_i} \, dz \right)^{1-\varepsilon_i} = 1 \]

\[ c_t(z) = \left( \frac{p_t(z)}{P_t^c} \right)^{-\varepsilon_i} C_t \]

Proof for cost minimization:

\[ \pi = P_t^c C_t - \int_0^1 p_t(z) c_t(z) \, dz \]

\[ \frac{d\pi}{dc_t(z)} = P_t^c \frac{\varepsilon_i}{\varepsilon_i - 1} C_t^{\frac{1}{\varepsilon_i}} \frac{1}{\varepsilon_i - 1} c_t(z)^{-1} - p_t(z) = 0 \]
B. Final Investment Good

\[ X_t = \frac{1}{\xi_t} \left( \int_0^1 x_t(z) \frac{\epsilon_t}{\epsilon_t - 1} \frac{\epsilon_t}{\epsilon_t - 1} d\zeta \right) \]

\[ P_t^x = \xi_t \left( \int_0^1 p_t(z)^{1-\epsilon_t} d\zeta \right)^{1-\epsilon_t} = \xi_t \]

\[ x_t(z) = \xi_t \left( \frac{p_t(z)}{P_t^c} \right)^{-\epsilon_t}, \quad X_t = \xi_t p_t(z)^{-\epsilon_t}, \quad X_t \]
C. Producers of Intermediate Inputs

Given: \( y_t(z) = F(k_t(z), n_t(z)) \)

*Capital* \( \rightarrow R_t \)

*Labor* \( \rightarrow W_t \)

*Aggregate Demand:*

\[ Y_t = C_t + \xi_t X_t \]

\[
\max \Pi_t(z) = p_t(z)y_t(z) - R_t k_t(z) - W_t n_t(z) \\
\text{s.t. } y_t(z) = c_t(z) + x_t(z) = p_t(z)^{-\epsilon_t} (C_t + \xi_t X_t) = p_t(z)^{-\epsilon_t} Y_t
\]
C. Producers of Intermediate Inputs

The first order condition with respect to capital:

\[ p_t(z)F_{n,t}(z) = \mu_t W_t \]
\[ p_t(z)F_{k,t}(z) = \mu_t R_t \]
\[ \mu_t = \frac{\varepsilon_t}{\varepsilon_t - 1} \]

Proof:

\[ \Pi_t(z) = \left(\frac{Y_t}{y_t(z)}\right)^{\frac{1}{\varepsilon_t}} y_t(z) - R_t k_t(z) - W_t n_t(z) = Y_t^{\frac{1}{\varepsilon_t}} y_t(z) - R_t k_t(z) - W_t n_t(z) \]
\[ \frac{\partial \Pi_t(z)}{\partial k_t(z)} = (1 - \frac{1}{\varepsilon_t}) \left(\frac{Y_t}{y_t(z)}\right)^{\frac{1}{\varepsilon_t}} F_{k,t}(z) - R_t = 0 \]
D. Household

\[
\max_{\pi_{t=0}^{\infty}} \beta^{t-t_0} V(C_t, N_t; \pi_t)
\]

s.t.

\[K_{t+1} = (1 - \delta)K_t + X_t\]

\[C_t + \xi_t X_t + B_{t+1} - (1 + r_t)B_t = \int_0^1 (W_t n_t(z) + R_t k_t(z) + \Pi_t(z))dz\]

\[N_t = \int_0^1 n_t(z)dz\]

\[K_t = \int_0^1 k_t(z)dz\]

The first order condition with respect to capital:

\[R_{t+1} = \xi_t (1 + r_{t+1}) - \xi_{t+1} (1 - \delta)\]

\[1 + r_{t+1} = \frac{V_c(C_t, N_t)}{\beta V_c(C_{t+1}, N_{t+1})}\]
Lagrange

\[ L = \sum_{t=t_0}^{\infty} \beta^{t-t_0} V(C_t, N_t; \chi_t) + \]

\[ \lambda_t \left[ K_{t+1} + (\delta - 1)K_t - \frac{1}{\xi_t} \left( \int_0^1 \Pi_t(z)dz + W_tN_t + R_tK_t + (1 + r_t)B_t - B_{t+1} - C_t \right) \right] \]

\[ \frac{\partial L}{\partial C_t} = \beta^{t-t_0} V_C(C_t, N_t) + \lambda_t / \xi_t = 0 \]

\[ \frac{\partial L}{\partial K_t} = \lambda_{t-1} + \lambda_t (\delta - 1 - R_t / \xi_t) = 0 \]
Hamilton

\[ H = \beta^{t-t_0} V(C_t, N_t; \chi_t) + \]

\[ \lambda_t \left[ -\delta K_t + \frac{1}{\xi_t} \left( \int_0^1 \Pi_t(z)dz + W_t N_t + R_t K_t + (1 + r_t)B_t - B_{t+1} - C_t \right) \right] \]

\[ \frac{\partial H}{\partial C_t} = \beta^{t-t_0} V_C(C_t, N_t) - \lambda_t / \xi_t = 0 \]

\[ \frac{\partial H}{\partial K_t} = \lambda_t (-\delta + R_t / \xi_t) = \lambda_{t-1} - \lambda_t \]
Bellman

\[ U(C_t, \{n_t(z)\}, X_t, K_{t+1}, B_{t+1}) = \max V(C_t, N_t, \chi_t) + \beta V(C_{t+1}, N_{t+1}, \chi_{t+1}) \]

\[
\frac{\partial U}{\partial K_{t+1}} = \frac{\partial V(C_t, N_t, \chi_t)}{\partial C_t} \frac{\partial C_t}{\partial K_{t+1}} + \beta \frac{\partial V(C_{t+1}, N_{t+1}, \chi_{t+1})}{\partial C_{t+1}} \frac{\partial C_{t+1}}{\partial K_{t+1}}
\]

\[
= -V_C(C_t, N_t)\xi_t + \beta V_C(C_{t+1}, N_{t+1})[R_{t+1} - \xi_{t+1}(\delta - 1)] = 0
\]
E. Equilibrium

\[ p_t(z) = P_t^c = 1, k_t(z) = K_t, n_t(z) = N_t, c_t(z) = C_t, \]
\[ x_t(z) = \xi_t X_t, y_t(z) = C_t + \xi_t X_t, Y_t = F(K_t, N_t) \]

\[ s_{L,t} = \frac{W_t N_t}{Y_t} = \frac{1}{\mu_t} \frac{W_t N_t}{W_t N_t + R_t K_t} \]
\[ s_{K,t} = \frac{R_t K_t}{Y_t} = \frac{1}{\mu_t} \frac{R_t K_t}{W_t N_t + R_t K_t} \]
\[ s_{\Pi,t} = \frac{\Pi_t}{Y_t} = 1 - \frac{1}{\mu_t} \]

Where \[ s_{L,t} + s_{K,t} + s_{\Pi,t} = 1 \]
F. The Production Function

\[ Y_t = F(K_t, N_t) = (\alpha_k (A_{K,t} K_t)^{\frac{\sigma-1}{\sigma}} + (1 - \alpha_k)(A_{N,t} N_t)^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}} \]

\[ F_{K,t} = \alpha_k A_{K,t}^{\frac{\sigma-1}{\sigma}} \left( \frac{Y_t}{K_t} \right)^{\frac{1}{\sigma}} = \mu_t R_t \]

\[ F_{N,t} = (1 - \alpha_k) A_{N,t}^{\frac{\sigma-1}{\sigma}} \left( \frac{Y_t}{N_t} \right)^{\frac{1}{\sigma}} = \mu_t W_t \]
G. The Labor Share

\[ 1 - s_{L,t} \mu_t = \alpha_k^\sigma \left( \frac{A_{K,t}}{\mu_t R_t} \right)^{\sigma-1} \]

\[ 1 - s_{L,t'} \mu_{t'} = \alpha_k^\sigma \left( \frac{A_{K,t'}}{\mu_{t'} R_{t'}} \right)^{\sigma-1} \]

\[ Z = \frac{Z_{t'}}{Z_t} - 1 \]

\[ \frac{1}{1 - s_L (1 + \hat{s}_L) \mu (1 + \mu)} = \left( \frac{1 + A_K}{(1 + \mu)(1 + R)} \right)^{\sigma-1} \]
IV. The Elasticity of Substitution

A. Relative Price of Investment

B. Markups

C. Capital-Augmenting Technological Progress

D. Skilled versus Unskilled Labor
A. Relative Price of Investment

\[ \frac{1}{1-s_L} \left( 1 - s_L (1 + \hat{s}_L) \mu (1 + \mu) \right) = \left( \frac{1 + A_K}{(1 + \mu)(1 + R)} \right)^{\sigma - 1} \]

\[ R_j = \xi_j \left( \frac{1}{\beta_j} - 1 + \delta_j \right) \]

Set \( \mu = 1, \mu = 0, \hat{A}_K = 0 \)

First-order approximation:

\[ \frac{s_{L,j}}{1-s_{L,j}} \hat{s}_{L,j} = \gamma + (\sigma - 1) \xi_j + u_j \]

\[ \hat{s}_{L,j} = \ln s_{L,j}, \xi_j = \ln \xi_j \]
A. Results

<table>
<thead>
<tr>
<th>Labor share</th>
<th>Investment price</th>
<th>$\hat{\sigma}$</th>
<th>Std. err.</th>
<th>90% Conf. interval</th>
<th>Obs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) KN Merged</td>
<td>PWT</td>
<td>1.25</td>
<td>0.08</td>
<td>[1.11,1.38]</td>
<td>58</td>
</tr>
<tr>
<td>(ii) KN Merged</td>
<td>WDI</td>
<td>1.29</td>
<td>0.07</td>
<td>[1.18,1.41]</td>
<td>54</td>
</tr>
<tr>
<td>(iii) OECD and UN</td>
<td>PWT</td>
<td>1.20</td>
<td>0.08</td>
<td>[1.06,1.34]</td>
<td>50</td>
</tr>
<tr>
<td>(iv) OECD and UN</td>
<td>WDI</td>
<td>1.31</td>
<td>0.06</td>
<td>[1.20,1.42]</td>
<td>47</td>
</tr>
<tr>
<td>(v) KLEMS 1</td>
<td>KLEMS</td>
<td>1.17</td>
<td>0.06</td>
<td>[1.06,1.27]</td>
<td>129</td>
</tr>
<tr>
<td>(vi) KLEMS 2</td>
<td>KLEMS</td>
<td>1.49</td>
<td>0.13</td>
<td>[1.28,1.70]</td>
<td>129</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td></td>
<td><strong>1.28</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A. Visualization
B. Markups

\[ \mu_j = \frac{1}{1-s_{\pi,j}} = \frac{1}{s_{L,j} + s_{K,j}} \]

\[ R_j = \xi_j \left( \frac{1}{\beta_j} - 1 + \delta_j \right) \]

\[ K_j = \frac{X_j}{\delta_j} \]

\[ s_{K,j} = \frac{R_j K_j}{Y_j} = \left( \frac{\xi_j X_j}{Y_j} \right) \left( \frac{1}{\beta_j} - 1 + \delta_j \right) \]

\[ \hat{\mu}_j = \frac{1}{\mu_j (s_{L,j} \hat{s}_{L,j} + s_{K,j} \hat{s}_{K,j})} \]

\[ \frac{s_{L,j} \mu_j}{1 - s_{L,j} \mu_j} \left( (1 + \hat{s}_{L,j})(1 + \hat{\mu}_j) - 1 \right) = \gamma + (\sigma - 1) \left( \hat{x}_j + \hat{\mu}_j \right) + u_j \]
### TABLE II

**Estimates of Elasticity of Substitution Allowing for Markups**

<table>
<thead>
<tr>
<th>Labor share</th>
<th>Investment price</th>
<th>Investment rate</th>
<th>( \hat{\sigma} )</th>
<th>Std. err.</th>
<th>90% Conf. interval</th>
<th>Obs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) KN Merged</td>
<td>PWT</td>
<td>Corporate</td>
<td>1.03</td>
<td>0.09</td>
<td>[0.87,1.19]</td>
<td>55</td>
</tr>
<tr>
<td>(ii) KN Merged</td>
<td>WDI</td>
<td>Corporate</td>
<td>1.29</td>
<td>0.08</td>
<td>[1.16,1.42]</td>
<td>52</td>
</tr>
<tr>
<td>(iii) OECD and UN</td>
<td>PWT</td>
<td>Corporate</td>
<td>1.24</td>
<td>0.11</td>
<td>[1.05,1.43]</td>
<td>46</td>
</tr>
<tr>
<td>(iv) OECD and UN</td>
<td>WDI</td>
<td>Corporate</td>
<td>1.43</td>
<td>0.08</td>
<td>[1.28,1.57]</td>
<td>44</td>
</tr>
<tr>
<td>(v) KN Merged</td>
<td>PWT</td>
<td>Total</td>
<td>1.11</td>
<td>0.11</td>
<td>[0.93,1.29]</td>
<td>54</td>
</tr>
<tr>
<td>(vi) KN Merged</td>
<td>WDI</td>
<td>Total</td>
<td>1.35</td>
<td>0.08</td>
<td>[1.22,1.49]</td>
<td>52</td>
</tr>
<tr>
<td>(vii) OECD and UN</td>
<td>PWT</td>
<td>Total</td>
<td>1.24</td>
<td>0.11</td>
<td>[1.06,1.43]</td>
<td>46</td>
</tr>
<tr>
<td>(viii) OECD and UN</td>
<td>WDI</td>
<td>Total</td>
<td>1.42</td>
<td>0.09</td>
<td>[1.27,1.56]</td>
<td>44</td>
</tr>
</tbody>
</table>

**Average** 1.26
C. Capital-Augmenting Technological Progress

\[ \frac{1}{1 - s_L \mu} (1 - s_L (1 + \hat{s}_L) \mu (1 + \mu)) = \left( \frac{1 + A_K}{(1 + \mu)(1 + R)} \right)^{\sigma - 1} \]

Set \( \mu = 1, \hat{\mu} = 0 \)

First-order approximation:

\[ \frac{s_{L,j}}{1 - s_{L,j}} \hat{s}_{L,j} = \gamma + (\sigma - 1) \hat{\xi}_j + (1 - \sigma) \hat{A}_{K,j} + u_j \]
C. Capital-Augmenting Technological Progress

\[ \tilde{\sigma} - \sigma = (1 - \sigma) \text{corr}(\hat{A}_{K,j}, \hat{\xi}_j) \frac{sd(\hat{A}_K)}{sd(\hat{\xi})} \]

\[ \tilde{\sigma} = 1.25 \]

\[ \sigma = 1.20 \]
D. Skilled versus Unskilled Labor

How to nest the three inputs: skilled labor, unskilled labor and the capital stock?

(1) \[ N_t = N_t(S_t, U_t) \]

\[ Y_t = \left( \phi \left( \phi_2 K_t ^{\rho - \rho^{-1} \frac{\rho - 1}{\rho}} + (1 - \phi_2 U_t) \frac{\rho - 1}{\rho} \right) + (1 - \phi) S_t ^{\frac{\sigma - 1}{\sigma}} \right)^{\frac{\sigma}{\sigma - 1}} \]

(2) \[ \frac{S_{L,j}}{1 - s_{L,j}} \hat{s}_{L,j} = \gamma + (\sigma - 1) \hat{s}_{J,j} + \kappa \left( \frac{U_{j}}{K_j} \right) + u_j \]

\[ Y_t = \left( \phi \left( \phi_2 K_t ^{\rho - \rho^{-1} \frac{\rho - 1}{\rho}} + (1 - \phi_2 S_t) \frac{\rho - 1}{\rho} \right) + (1 - \phi) U_t ^{\frac{\sigma - 1}{\sigma}} \right)^{\frac{\sigma}{\sigma - 1}} \]

(3) \[ \frac{S_{L,j}}{1 - s_{L,j}} \hat{s}_{L,j} = \gamma + (\sigma - 1) \hat{s}_{J,j} + \kappa \left( \frac{S_j}{K_j} \right) + u_j \]
D. Results

<table>
<thead>
<tr>
<th>Labor share</th>
<th>Nested input with capital</th>
<th>$\hat{\sigma}$</th>
<th>Std. err.</th>
<th>90% Conf. interval</th>
<th>Obs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) KLEMS 1</td>
<td>High skill</td>
<td>1.23</td>
<td>0.08</td>
<td>[1.11,1.36]</td>
<td>100</td>
</tr>
<tr>
<td>(ii) KLEMS 1</td>
<td>Middle and low skill</td>
<td>1.19</td>
<td>0.08</td>
<td>[1.05,1.33]</td>
<td>100</td>
</tr>
<tr>
<td>(iii) KLEMS 1</td>
<td>Low skill</td>
<td>1.19</td>
<td>0.09</td>
<td>[1.04,1.34]</td>
<td>100</td>
</tr>
<tr>
<td>(iv) KLEMS 2</td>
<td>High skill</td>
<td>1.34</td>
<td>0.16</td>
<td>[1.07,1.60]</td>
<td>100</td>
</tr>
<tr>
<td>(v) KLEMS 2</td>
<td>Middle and low skill</td>
<td>1.31</td>
<td>0.17</td>
<td>[1.03,1.60]</td>
<td>100</td>
</tr>
<tr>
<td>(vi) KLEMS 2</td>
<td>Low skill</td>
<td>1.31</td>
<td>0.18</td>
<td>[1.02,1.61]</td>
<td>100</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td></td>
<td><strong>1.26</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
V. The Decline in the Labor Share

<table>
<thead>
<tr>
<th></th>
<th>CD</th>
<th>Ces</th>
<th>CD</th>
<th>Ces</th>
<th>CD</th>
<th>Ces</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) Labor share (percentage points)</td>
<td>(\hat{\xi})</td>
<td>-2.6</td>
<td>-3.1</td>
<td>-2.6</td>
<td>-3.1</td>
<td>-4.9</td>
</tr>
<tr>
<td>(ii) Capital share (percentage points)</td>
<td>0.0</td>
<td>2.6</td>
<td>-1.9</td>
<td>-2.4</td>
<td>-1.9</td>
<td>-0.1</td>
</tr>
<tr>
<td>(iii) Profit share (percentage points)</td>
<td>0.0</td>
<td>0.0</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
</tr>
<tr>
<td>(iv) Consumption</td>
<td>18.1</td>
<td>20.1</td>
<td>-5.2</td>
<td>-5.4</td>
<td>10.7</td>
<td>12.4</td>
</tr>
<tr>
<td>(v) Nominal investment</td>
<td>18.1</td>
<td>30.8</td>
<td>-11.1</td>
<td>-12.7</td>
<td>3.7</td>
<td>11.9</td>
</tr>
<tr>
<td>(vi) Labor input</td>
<td>0.0</td>
<td>-1.4</td>
<td>-3.2</td>
<td>-2.9</td>
<td>-3.2</td>
<td>-4.2</td>
</tr>
<tr>
<td>(vii) Capital input</td>
<td>51.6</td>
<td>67.8</td>
<td>-11.1</td>
<td>-12.7</td>
<td>33.2</td>
<td>43.6</td>
</tr>
<tr>
<td>(viii) Output</td>
<td>18.1</td>
<td>22.8</td>
<td>-6.3</td>
<td>-6.8</td>
<td>9.4</td>
<td>12.3</td>
</tr>
<tr>
<td>(ix) Wage</td>
<td>18.1</td>
<td>19.2</td>
<td>-8.2</td>
<td>-8.2</td>
<td>7.1</td>
<td>7.7</td>
</tr>
<tr>
<td>(x) Rental rate</td>
<td>-22.1</td>
<td>-22.1</td>
<td>0.0</td>
<td>0.0</td>
<td>-22.1</td>
<td>-22.1</td>
</tr>
<tr>
<td>(xi) Capital-to-output</td>
<td>28.4</td>
<td>36.6</td>
<td>-5.2</td>
<td>-6.4</td>
<td>21.8</td>
<td>27.9</td>
</tr>
<tr>
<td>(xii) Welfare equivalent consumption</td>
<td>18.1</td>
<td>22.1</td>
<td>-3.0</td>
<td>-3.4</td>
<td>13.2</td>
<td>15.8</td>
</tr>
</tbody>
</table>
VI. Conclusion

- Offer an explanation for the decline of global labor share
- Estimate the shape of the production function
- Explore the macroeconomic and welfare implications
Implication

How does $S$ change with $\mu, A, R$

\[ \left( \frac{1}{1-S_L \mu} \right) \left[ 1 - S_L \left( 1 + \hat{S} \right) \mu (1 + \hat{\mu}) \right] = \left[ \frac{1+\hat{A}}{(1+\hat{\mu})(1+\hat{R})} \right]^{\sigma-1} \]

$\sigma = 1$

\[ \hat{S} = \frac{1}{1 + \hat{\mu}} - 1 \]

$\sigma > 1$

\[ \hat{S} = \frac{1-(1-S_L \mu) \left[ \frac{1+\hat{A}}{(1+\hat{\mu})(1+\hat{R})} \right]^{\sigma-1}}{S_L \mu (1+\hat{\mu})} - 1 \]

\[ \frac{d\hat{S}}{d\hat{\mu}} = \frac{\sigma(1-S_L \mu) \left[ \frac{1+\hat{A}}{(1+\hat{\mu})(1+\hat{R})} \right]^{\sigma-1}}{S_L \mu (1+\hat{\mu})^2} \]

\[ \frac{d\hat{S}}{d\hat{\mu}} = 0 \Rightarrow \hat{\mu} = \frac{1+\hat{A}}{1+\hat{R}} \left[ \frac{1}{\sigma(1-S_L \mu)} \right]^{-1} \]
Thank you