推导：

预算约束：$D_t(i)=(1+r_{t-1})D_{t-1}(i)-\frac{1}{2}Y+C_t(i)$.

同时 $(1+r_t)D_t(i)\le D^{high}$.

构造 $L=E_t\sum_{t=0}^{\infty}\beta^i[\log C_t(i)+\lambda_t(i)(\frac{1}{2}Y+(1+r_{t-1})D_{t-1}(i)-D_t(i)-C_t(i))]$

$$\frac{\partial L}{\partial D_t(i)}=-\lambda_t(i)+E_t[\lambda_{t+1}(i)(1+r_t)]=0 \quad ①$$

$$\frac{\partial L}{\partial C_t(i)}=\frac{\beta^i}{C_t(i)}-\lambda_t(i)=0 \qquad \lambda_t(i)=\frac{\beta^i}{C_t(i)}$$ 代入①得

$$-\frac{\beta^i}{C_t(i)}+E_t\left[\frac{\beta^{i+1}}{C_{t+1}(i)}(1+r_t)\right]=0.$$

$$\therefore \frac{1}{C_t(i)}=\beta^i E_t\left[\frac{1+r_t}{C_{t+1}(i)}\right]$$

稳态下：$C_t(i)=C_{t+1}(i)$ $r_t=r$.

故欧拉方程：$\frac{1}{C(i)}=\beta^i\frac{1+r}{C(i)}$

整理得 $r=\frac{1}{\beta}-1$.

Ⅲ、储户的最优消费决策满足的欧拉方程.

$$C_L^S = (1+r_S)\beta C_S^S \quad (\text{assume: } L\text{与}S\text{相距一个周期})$$

证：最优化： $\max \sum_{t=0}^{\infty} \beta^t \ln(C_t^S)$

约束： $C_t^S + D_t = Y^S + (1+r_S) D_{t-1}$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \ln(C_t^S) + \lambda_t [Y^S + (1+r_S) D_{t-1} - C_L^S - D_t]$$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial C_t^S} = \beta^t \cdot \frac{1}{C_t} - \lambda_t = 0 \\ \lambda_t = \beta^t \frac{1}{C_t^S} \\ \frac{\partial \mathcal{L}}{\partial D_t} = -\lambda_t + \lambda_{t+1}(1+r_t) = 0 \\ \lambda_{t+1} = \lambda_t \frac{1}{1+r_S} \end{cases}$$

$$\Rightarrow \beta^{t+1} \frac{1}{C_{t+1}^S} = \beta^t \frac{1}{C_t^S} \frac{1}{1+r_S}$$

$$\therefore \frac{C_{t+1}^S}{C_t^S} = \frac{C_L^S}{C_S^S} = \beta(1+r_S)$$

$$\Rightarrow C_L^S = (1+r_S)\beta C_S^S$$

IV、saver 最优消费决策Euler equation (含P)：

$$\frac{1}{C_t^s} = (1+i_t)\beta E_t \frac{1}{C_{t+1}^s}\frac{P_t}{P_{t+1}}$$

证： $\max U = \max \sum_{t=0}^{\infty} \beta^t \ln(C_t^s)$

约束： $C_t^s + P_t D_t = Y_t^s + (1+r_s) P_{t-1} D_{t-1}$

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \ln(C_t^s) + \sum_{t=0}^{\infty} \lambda_t (Y_t^s + (1+r_s) P_{t-1} D_{t-1} - P_t C_t^s - P_t D_t)$$

$$\frac{\partial \mathcal{L}}{\partial C_t^s} = \beta^t \frac{1}{C_t^s} - \lambda_t P_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial D_t} = -\lambda_t P_t + \lambda_{t+1} P_{t+1} (1+r_s) = 0$$

$$\Rightarrow \frac{1}{C_t^s} = \beta (1+r_s) E_t \left[\frac{P_{t+1}}{P_t} \frac{1}{C_{t+1}^s}\right]_{whole}$$

若考虑名义利率 i_t，则：

$$\frac{1}{C_t^s} = (1+i_t)\beta E_t \left[\frac{1}{C_{t+1}^s} \frac{P_t}{P_{t+1}}\right]$$

家庭

对于家庭，面临的决策

$$E_0 \sum_{t=0}^{\infty} \beta^{(i)t} \left[u^i(C_t(i)) - v^i(h_t(i)) \right] \text{ where } i = s \text{ or } b$$

s.t.

$$B_t(i) = (1 + i_{t-1}) B_{t-1}(i) - W_t P_t h_t(i) - \int_0^1 \Pi_t(i) + P_t C_t(i) - T_t(i)$$

$$(1 + r_t) \frac{B_t(i)}{P_t} \leq D_t(i)$$

拉格朗日函数

$$\begin{aligned} \mathcal{L}_0(i) \quad = \quad & E_0 \sum_{t=0}^{\infty} \beta^{(i)t} \{ [u^i(C_t(i)) - v^i(h_t(i))] \\ & + \phi_{1t}(i)[B_t(i) - (1 + i_{t-1})B_{t-1}(i) + W_t P_t h_t(i) + \int_0^1 \Pi_t(i) - P_t C_t(i) - T_t(i)] \\ & + \phi_{2t}(i)[(1 + r_t)\frac{B_t(i)}{P_t} - D_t(i)] \} \end{aligned}$$

一阶条件

$$\frac{\partial \mathcal{L}_t(i)}{\partial C_t(i)} = u_c^i(C_t(i)) - \phi_{1t}(i) P_t = 0$$

$$\frac{\partial \mathcal{L}_t(i)}{\partial h_t(i)} = -v_h^i(h_t(i)) + P_t W_t \phi_{1t}(i) = 0$$

$$\frac{\partial \mathcal{L}_t(i)}{\partial B_t(i)} = \phi_{1t}(i) - \beta E_t \phi_{1t+1}(i)(1 + i_t) + \phi_{2t}(i) \frac{(1 + r_t)}{P_t} = 0$$

互补松弛条件

$$\phi_{2t}(i) \geq 0, \quad D_t(i) \geq (1 + r_t)\frac{B_t(i)}{P_t}, \qquad \phi_{2t}(i)[(1 + r_t)\frac{B_t(i)}{P_t} - D_t(i)] = 0$$

可得总体需求函数

$$c_t(j) = C_t(\frac{p_t(j)}{P_t})^{-\theta}$$

AS曲线推导

$$E_t\sum_{t=0}^{\infty}\phi_t^s[(1-\tau)p_t(j)y_t(j)-W_tP_th_t(j)]$$

s.t.

$$y_t(j) = Y_t(\frac{p_t(j)}{P_t})^{-\theta}$$
$$y_t(j) = h_t(j)$$

（1）模型设定

- 灵活定价企业（λ比例）：定价为 $p_t(1)$；
- 粘性定价企业：价格刚性，沿用前一期价格 $p_t(2)=p_{t-1}(2)$ 。
- 需求函数：差异化商品需求为 $c_t(j)=C_t(\frac{p_t(j)}{P_t})^{-\theta}$ ，总价格指数为 $P_t=\left(\int_0^1 p_t(j)^{1-\theta}dj\right)^{\frac{1}{1-\theta}}$

（2）灵活价格企业的最优定价：自由设定价格，灵活价格企业最大化预期利润的贴现值，最优价格满足：

$$(1-\tau)\frac{p_t(1)}{P_t}=\frac{\theta}{\theta-1}W_t$$

稳态下 $(1-\tau)\frac{\theta-1}{\theta}=1$ ，简化为 $p_t(1)=W_tP_t$

对数形式：

$$\log p_t(1)=\log P_t+\hat{W}_t$$

AS曲线推导

（3）价格指数的对数线性化

总价格指数： $P_t = \left[\lambda p_t(1)^{1-\theta} + (1-\lambda)p_t(2)^{1-\theta}\right]^{\frac{1}{1-\theta}}$

对粘性价格企业， $\log p_t(2) = \log P_{t-1} + \hat{W}_{t-1}$

价格指数变化为： $\log P_t = \lambda \log p_t(1) + (1-\lambda)\log p_t(2)$

代入表达式，整理得

$$\pi_t - E_{t-1}\pi_t = \frac{\lambda}{1-\lambda}\hat{W}_t$$

（4）代入劳动供给条件 $\hat{W}_t = \omega\hat{Y}_t + \sigma^{-1}\hat{C}_t$ **，利用线性化的总资源约束消去消费** $\hat{C}_t = \hat{Y}_t - \hat{G}_t$ **，则有AS曲线**

$$\pi_t = \kappa\hat{Y}_t - \kappa\psi\hat{G}_t + E_{t-1}\pi_t$$

其中， $\kappa = \frac{\lambda}{1-\lambda}\left(\omega + \sigma^{-1}\right)$， $\psi = \frac{\sigma^{-1}}{\omega+\sigma^{-1}}$

$$\pi_t = \kappa\hat{Y}_t + E_{t-1}\pi_t$$

消费者消费函数推导

2. 借款者的消费函数

- **最大化自身效用，收到预算约束和借贷约束，构建拉格朗日函数，并求一阶条件，进行稳态分析**

$$\bar{C}^b = \bar{W}\bar{h}^b - \bar{r}\bar{D}^b - \bar{T}^b \qquad (1+\bar{r})\frac{\bar{B}^b}{P} = \bar{D}^b \qquad \bar{r} = \beta^{-1} - 1$$

- **预算约束线性化，得到**

$$\hat{B}_t^b = (1+\bar{r})\hat{B}_{t-1}^b + \beta\hat{D}_t - \hat{D}_{t-1} + \gamma_D\pi_t - \gamma_D\beta(i_t - E_t\pi_{t+1} - \bar{r}) - \hat{T}_t^b;$$

- **消费一阶条件，对数线性化**

$$\hat{C}_t^b = \sigma^b\left(\hat{W}_t + \hat{h}_t^b\right)$$

- **结合劳动市场均衡条件，并利用资源约束，得到：**

$$\hat{C}_t^b = \hat{I}_t^b + \beta\hat{D}_t - \hat{D}_{t-1} + \gamma_D\pi_t - \gamma_D\beta(i_t - E_t\pi_{t+1} - \bar{r}) - \hat{T}_t^b$$

稳态

$$\frac{v_h^b(\bar{h}^b)}{u_c^b(\bar{C}^b)} = W$$

$$\frac{v_h^s(\bar{h}^s)}{u_c^s(\bar{C}^s)} = W$$

$$W = (1-\tau)\frac{\theta - 1}{\theta}$$

$$\bar{C}^b = -r\bar{D} + \bar{W}\bar{h}^b - \bar{T}^b$$

$$\bar{Y} = \chi_s \bar{C}^s + \chi_b \bar{C}^b + \bar{G}$$

$$\chi_b \frac{\bar{D}}{1+r} + \bar{b}^g = -\chi_s \bar{b}^s$$

$$\bar{Y} = \chi_s \bar{h}^s + \chi_b \bar{h}^b$$

$$1 + \bar{r} = \beta^{-1}$$

线性对数化

人均消费 $C_t = \chi^s C_t^s + (1-\chi^s)C_t^b$

生产 $h_t = \chi^s h_t^s + (1-\chi^s)h_t^b$

总资源约束 $Y_t = C_t + G_t$

对数线性化得到， $\hat{C}_t^b = \hat{I}_t^b + \beta\hat{D}_t - \hat{D}_{t-1} + \gamma_D\pi_t - \gamma_D\beta(i_t - E_t\pi_{t+1} - \bar{r}) - \hat{T}_t^b$

储蓄者的欧拉方程 $u_c^s(C_t^s) = \beta(1+i_t)E_t u_c^s(C_{t+1}^s)\frac{P_t}{P_{t+1}}$

对数线性化， $\hat{C}_t^s = E_t\hat{C}_{t+1}^s - \sigma(i_t - E_t\pi_{t+1} - \bar{r})$

欧拉方程推导

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta_s^t \{[u^s(C_t^s) - v^s(h_t^s)] + \lambda_t [(1 + i_{t-1})B_{t-1} + W_t h_t^s + D_t - P_t C_t^s - B_t]\}$$

$$\frac{\partial \mathcal{L}}{\partial C_t^s} = E_0 \left[\beta_s^t u_c^s(C_t^s) - \lambda_t P_t\right] = 0 \qquad \lambda_t = \beta_s^t \frac{u_c^s(C_t^s)}{P_t} \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = E_0 \left[-\lambda_t + \beta_s^{t+1} \lambda_{t+1}(1 + i_t) \cdot \beta_s^{-1}\right] = 0 \qquad -\lambda_t + \beta_s E_t \left[\lambda_{t+1}(1 + i_t)\right] = 0 \quad (2)$$

$$\lambda_t = \beta_s^t \frac{u_c^s(C_t^s)}{P_t}, \quad \lambda_{t+1} = \beta_s^{t+1} \frac{u_c^s(C_{t+1}^s)}{P_{t+1}} \qquad -\left(\beta_s^t \frac{u_c^s(C_t^s)}{P_t}\right) + \beta_s E_t \left[\left(\beta_s^{t+1} \frac{u_c^s(C_{t+1}^s)}{P_{t+1}}\right)(1 + i_t)\right] = 0$$

$$u_c^s(C_t^s) = \beta_s (1 + i_t) E_t \left[u_c^s(C_{t+1}^s) \frac{P_t}{P_{t+1}}\right]$$

7.1.

消费者跨期优化：

1. 储蓄者消费欧拉方程 $\frac{1}{C_t^s} = (1+r_t)\beta E_t \frac{1}{C_{t+1}^s}$

对数线性化（稳态附近）： $\hat{C}_t^s = E_t \hat{C}_{t+1}^s - \sigma(i_t - E_t \pi_{t+1} - \bar{r})$

2. 借款人短期消费决策：

$$\hat{C}_s^b = \hat{I}_s^b - \hat{D} + \gamma_D \pi_s - \gamma_D \beta (i_s - \bar{r}) - \hat{T}_s^b$$

$(\gamma_D = \bar{D}/\bar{Y})$

内生债务限额：Lagrange 约束

$$D_{bt} = (1-\gamma^L) D^2 + (1+r_t)\gamma^L E_t \sum_{j=1}^{\infty} R_{t,t+j} (I_{t+j}^b - T_{t+j}^b)$$

$$L = \sum_{t=0}^{\infty} \beta^t \{ u(C_t) - v(h_t) + \lambda_t [D_{bt} - (1+r_t)\frac{B_t}{P_t}] \}$$

F.O.C $\quad \frac{\partial L}{\partial B_t} = 0 \Rightarrow \lambda_t = \beta E_t \lambda_{t+1} (1+r) + \dots$

7.2.

欧拉方程：储蓄者跨期消费决策

$$\frac{1}{C_t^s} = (1+i_t)\beta E_t \frac{1}{C_{t+1}^s} \frac{P_t}{P_{t+1}}$$

$$\hat{C}_t^s = E_t \hat{C}_{t+1}^s - \sigma(i_t - E_t \pi_{t+1} - \bar{r})$$

Lagrange 推导企业边际成本

利润最大化： $\max_{h_t^b} P_t Y_t - W_t h_t^b \quad s.t \quad Y_t = A h_t^b$

h_t^b 为借款人劳动供给。

$$L = P_t A h_t^b - W_t h_t^b + \mu_t (\bar{h} - h_t^b)$$

（μ_t 为劳动供给上限的 Lagrange 乘数）

F.O.C $\frac{\partial L}{\partial n_t}=0 \Rightarrow AP_t = W_t + \mu_t$

$$MC_t = \frac{W_t}{A} + \frac{\mu_t}{A}$$

7.3.

扩展效用函数

$$u^i(C_t(i)) + d^i(K_t(i)),\quad K_t(i) = I_t(i) + (1-\delta)K_{t-1}(i)$$

$(0 \le \delta \le 1)$

F.O.C. $\hat{C}_i^i = (1-\beta_i(1-\delta))\frac{\partial c}{\partial K}\hat{K}_t^i + \beta_i(1-\delta)E_t\hat{C}_{t+1}^i$

生产性投资: $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$ 调节成本 $\gamma(\frac{I_t}{I})$ $\gamma''(\cdot)>0$

企业利润最大化: $\max_{I_t} E_t \sum_{t=0}^{\infty} \beta^t [\pi_t - \gamma(\frac{I_t}{I})]$

$$\hat{I}_t = \frac{\beta(1-\delta)}{1+\beta(1-\delta)\sigma_I} E_t \hat{I}_{t+1} + \frac{\sigma_I}{1+\beta(1-\delta)\sigma_I}\hat{q}_t$$

借款人预算约束. $L = \sum_{t=0}^{\infty}\beta^t [u(C_t^b) + d(K_t^b) - \lambda_t((1+r_t)\frac{B_t}{P_t} - D_{bt})]$

$$\frac{MU_{C_t^b}}{MU_{I_t^b}} = \frac{1}{1+r_t-\delta}$$ 最优支出分配条件.

7.4

跨期消费欧拉方程.

$$\hat{C}_t^s = E_t\hat{C}_{t+1}^s - \sigma(i_t - E_t\pi_{t+1} - r_t^e)$$

动态IS: $\hat{Y}_t = E_t\hat{Y}_{t+1} - \sigma(i_t - E_t\pi_{t+1} - r_t^e) + \frac{1}{1-x_b\mu}\hat{G}_t$

($\hat{Y}_t = x_s\hat{C}_t^s + x_b\hat{C}_t^b + \hat{G}_t$,结合借款人消费约束与储蓄者

欧拉方程（消去 $\hat{C}^s_t$ 后得）

借款人债务约束 Lagrange 构造.

$$L = \sum_{t=0}^{\infty} \beta^t \left[u(\hat{C}^b_t) - \lambda_t \left((1+r_t)\frac{B_t}{P_t} - D_{bt} \right) \right]$$

F.O.C $\hat{D}_t = \beta E_t \hat{D}_{t+1} + \hat{Y}_t - \hat{C}^b_t$

财政政策乘数设定.

$$\hat{Y}_t = \frac{1}{1-\chi_b\mu} \hat{G}_t + \underbrace{\sigma E_t \hat{Y}_{t+1}}_{\text{预期项}}$$